

ON DOUBLY PERIODIC PHASES

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ABSTRACT. Every meromorphic function on \mathbb{C} with doubly periodic phase is equal to an elliptic function multiplied by a meromorphic function determined by the periods.

The purpose of this note is to completely characterize the meromorphic functions on \mathbb{C} that have doubly periodic phases. In particular, our result shows that the characterization suggested by G. Semmler and E. Wegert in [4] is too narrow. The phase of a function $f(z)$ defined on an open set $D \subset \mathbb{C}$ with values in $\widehat{\mathbb{C}}$ is the function $\frac{f(z)}{|f(z)|}$ defined on $\{z \in D : f(z) \in \widehat{\mathbb{C}} \setminus \{0, \infty\}\}$ with values in \mathbb{T} .

We recall the definition of the Weierstrass sigma-function ([6]). If $p_1, p_2 \in \mathbb{C}$ are linearly independent over \mathbb{R} , we denote by L the \mathbb{Z} -module generated by p_1, p_2 and we put $\omega = \frac{p_2}{p_1}$. Since the series

$$\sum_{\lambda \in L \setminus \{0\}} \frac{1}{|\lambda|^3}$$

is convergent, the Weierstrass product ([5])

$$z \prod_{\lambda \in L \setminus \{0\}} e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}} \left(1 - \frac{z}{\lambda}\right)$$

defines an entire function $\sigma(z)$ whose set of zeros is precisely L , and which is known as the Weierstrass sigma-function. The following transformation property is probably well-known but we include a proof, for completeness.

Lemma. *For every $\xi_0 \in \mathbb{C}$ and $j \in \{1, 2\}$ the Weierstrass sigma-function satisfies*

$$\frac{\sigma(z)}{\sigma(-\xi_0 + z)} \cdot \frac{\sigma(-\xi_0 + z + p_j)}{\sigma(z + p_j)} = e^{v_j},$$

where

$$v_j = -\frac{3\xi_0}{p_j} + \xi_0 p_j^2 \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{\lambda(\lambda + p_j)^2}.$$

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Proof. For every $\xi_0 \in \mathbb{C}$ and $j \in \{1, 2\}$ we have

$$\begin{aligned} \frac{\sigma(z)}{\sigma(-\xi_0 + z)} &= \frac{z}{-\xi_0 + z} \prod_{\lambda \in L \setminus \{0\}} e^{\frac{\xi_0}{\lambda} - \frac{\xi_0^2}{2\lambda^2} + \frac{\xi_0 z}{\lambda^2}} \frac{\lambda - z}{\lambda + \xi_0 - z} \\ &= \frac{z(z + p_j) e^{-\frac{\xi_0}{p_j} - \frac{\xi_0^2}{2p_j^2} + \frac{\xi_0 z}{p_j^2}}}{(-\xi_0 + z)(-\xi_0 + z + p_j)} \\ &\quad \cdot \prod_{\lambda \in L \setminus \{0, -p_j\}} e^{\frac{\xi_0}{\lambda} - \frac{\xi_0^2}{2\lambda^2} + \frac{\xi_0 z}{\lambda^2}} \frac{\lambda - z}{\lambda + \xi_0 - z} \end{aligned}$$

and

$$\begin{aligned} \frac{\sigma(z + p_j)}{\sigma(-\xi_0 + z + p_j)} &= \frac{z + p_j}{-\xi_0 + z + p_j} \prod_{\lambda \in L \setminus \{0\}} e^{\frac{\xi_0}{\lambda} - \frac{\xi_0^2}{2\lambda^2} + \frac{\xi_0(z + p_j)}{\lambda^2}} \frac{\lambda - p_j - z}{\lambda - p_j + \xi_0 - z} \\ &= \frac{z + p_j}{-\xi_0 + z + p_j} \\ &\quad \cdot \prod_{\lambda \in L \setminus \{-p_j\}} e^{\frac{\xi_0}{\lambda + p_j} - \frac{\xi_0^2}{2(\lambda + p_j)^2} + \frac{\xi_0(z + p_j)}{(\lambda + p_j)^2}} \frac{\lambda - z}{\lambda + \xi_0 - z} \\ &= \frac{z(z + p_j) e^{\frac{2\xi_0}{p_j} - \frac{\xi_0^2}{2p_j^2} + \frac{\xi_0 z}{p_j^2}}}{(-\xi_0 + z)(-\xi_0 + z + p_j)} \\ &\quad \cdot \prod_{\lambda \in L \setminus \{0, -p_j\}} e^{\frac{\xi_0}{\lambda + p_j} - \frac{\xi_0^2}{2(\lambda + p_j)^2} + \frac{\xi_0(z + p_j)}{(\lambda + p_j)^2}} \frac{\lambda - z}{\lambda + \xi_0 - z} \end{aligned}$$

therefore

$$\begin{aligned} &\frac{\sigma(z)}{\sigma(-\xi_0 + z)} \cdot \frac{\sigma(-\xi_0 + z + p_j)}{\sigma(z + p_j)} \\ &= e^{-\frac{3\xi_0}{p_j} + \sum_{\lambda \in L \setminus \{0, -p_j\}} \left(\frac{\xi_0}{\lambda} - \frac{\xi_0}{\lambda + p_j} - \frac{\xi_0^2}{2\lambda^2} + \frac{\xi_0^2}{2(\lambda + p_j)^2} + \frac{\xi_0 z}{\lambda^2} - \frac{\xi_0(z + p_j)}{(\lambda + p_j)^2} \right)} \\ &= e^{u_j z + v_j - \frac{1}{2}\xi_0 u_j}, \end{aligned}$$

where

$$u_j = \xi_0 \sum_{\lambda \in L \setminus \{0, -p_j\}} \left(\frac{1}{\lambda^2} - \frac{1}{(\lambda + p_j)^2} \right).$$

For $\omega \in \mathbb{C} \setminus \mathbb{R}$, we define $f_1(\omega)$ by

$$\begin{aligned} p_1^2 u_1 &= p_1^2 \xi_0 \sum_{\lambda \in L \setminus \{0, -p_1\}} \left(\frac{1}{\lambda^2} - \frac{1}{(\lambda + p_1)^2} \right) \\ &= \xi_0 \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0), (-1,0)\}} \left(\frac{1}{(m + n\omega)^2} - \frac{1}{(m + 1 + n\omega)^2} \right) \\ &=: f_1(\omega) \end{aligned}$$

and notice that

$$\begin{aligned} p_2^2 u_2 &= p_2^2 \xi_0 \sum_{\lambda \in L \setminus \{0, -p_2\}} \left(\frac{1}{\lambda^2} - \frac{1}{(\lambda + p_2)^2} \right) \\ &= \xi_0 \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0), (0,-1)\}} \left(\frac{1}{(\frac{m}{\omega} + n)^2} - \frac{1}{(\frac{m}{\omega} + n + 1)^2} \right) \\ &= f_1\left(\frac{1}{\omega}\right). \end{aligned}$$

It now suffices to show that $f_1(\omega) = 0$ for every $\omega \in \mathbb{C} \setminus \mathbb{R}$:

$$\begin{aligned} f_1(\omega) &= \xi_0 \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{m \in \mathbb{Z}} \left(\frac{1}{(m + n\omega)^2} - \frac{1}{(m + 1 + n\omega)^2} \right) \\ &\quad + \xi_0 \sum_{m \in \mathbb{Z} \setminus \{-1, 0\}} \left(\frac{1}{m^2} - \frac{1}{(m + 1)^2} \right) \\ &= 0. \end{aligned}$$

□

Theorem. *Let $f(z)$ be a nonconstant function meromorphic on \mathbb{C} . If the phase of $f(z)$ is doubly periodic with primitive periods p_1, p_2 and $\omega = \frac{p_2}{p_1} \in \mathbb{C} \setminus \mathbb{R}$, then there exists an elliptic function $g(z)$ with periods p_1, p_2 such that*

$$f(z) = e^{az} g(z) \frac{\sigma(z)}{\sigma(-\xi_0 + z)},$$

where $\xi_0 \in \mathbb{C}$, and $a \in \mathbb{C}$ satisfies $\Im(ap_j) = \Im(v_j) + 2m_j\pi$, $j \in \{1, 2\}$, for some $m_1, m_2 \in \mathbb{Z}$.

Proof. We shall use the observation ([4]) that if the phase of a meromorphic function $f(z)$ on \mathbb{C} has a period p , then $f(z + p) = e^\alpha f(z)$ for some constant $\alpha \in \mathbb{R}$. Let α_1, α_2 be two real numbers such that $f(z + p_1) = e^{\alpha_1} f(z)$ and $f(z + p_2) = e^{\alpha_2} f(z)$. Let F denote the parallelogram with the two vectors p_1, p_2 as adjacent sides. The function

$\frac{f'(z)}{f(z)}$ is obviously elliptic with periods p_1, p_2 and its integral over F is therefore 0. In particular, $f(z)$ has the same number of zeros and poles inside F . Let Ξ be the multiset (we take into consideration the multiplicities that occur) consisting of all zeros of $f(z)$ inside F and Γ be the multiset consisting of all poles of $f(z)$ inside F . We then have

$$\begin{aligned}
\sum_{\xi \in \Xi} \xi - \sum_{\gamma \in \Gamma} \gamma &= \frac{1}{2\pi i} \int_F \frac{zf'(z)}{f(z)} dz \\
&= \frac{1}{2\pi i} \int_0^{p_1} (z - (z + p_2)) \frac{f'(z)}{f(z)} dz \\
&\quad - \frac{1}{2\pi i} \int_0^{p_2} (z - (z + p_1)) \frac{f'(z)}{f(z)} dz \\
&= -\frac{p_2}{2\pi i} \int_0^{p_1} d \log f(z) + \frac{p_1}{2\pi i} \int_0^{p_2} d \log f(z) \\
&= -\frac{p_2}{2\pi i} (\alpha_1 + 2n_1\pi i) + \frac{p_1}{2\pi i} (\alpha_2 + 2n_2\pi i) \\
&\equiv -\xi_0 \pmod{L}
\end{aligned}$$

for some $n_1, n_2 \in \mathbb{Z}$, where ξ_0 is the unique number inside F that is congruent to $\frac{\alpha_1 p_2 - \alpha_2 p_1}{2\pi i}$ modulo L . The congruence above is equivalent to

$$\sum_{\xi \in \{\xi_0\} \cup \Xi} \xi - \sum_{\gamma \in \{0\} \cup \Gamma} \gamma \equiv 0 \pmod{L}$$

hence, by Abel's theorem ([1, 2, 3]), there exists an elliptic function $g(z)$ with periods p_1, p_2 such that its multisets of zeros and poles inside F are $\{\xi_0\} \cup \Xi$ and $\{0\} \cup \Gamma$, respectively. The sets of zeros and poles of the meromorphic function $\frac{g(z)}{f(z)}$ are then $\xi_0 + L$ and L , respectively. Since the meromorphic function $\frac{\sigma(-\xi_0 + z)}{\sigma(z)}$ has exactly the same zeros and poles as $\frac{g(z)}{f(z)}$, there exists an entire function $h(z)$ such that

$$f(z) = e^{h(z)} g(z) \frac{\sigma(z)}{\sigma(-\xi_0 + z)}.$$

The conditions $f(z + p_j) = e^{\alpha_j} f(z)$, $j \in \{1, 2\}$, now imply

$$e^{h(z+p_j)} \frac{\sigma(z+p_j)}{\sigma(-\xi_0 + z + p_j)} = e^{\alpha_j + h(z)} \frac{\sigma(z)}{\sigma(-\xi_0 + z)}$$

hence, using the Lemma,

$$\begin{aligned} e^{h(z+p_j)-h(z)-\alpha_j} &= \frac{\sigma(z)}{\sigma(-\xi_0+z)} \cdot \frac{\sigma(-\xi_0+z+p_j)}{\sigma(z+p_j)} \\ &= e^{v_j}. \end{aligned}$$

The function $h'(z)$ is therefore doubly periodic and has no poles, hence $h'(z)$ is constant. So $h(z) = az + b$ for some $a, b \in \mathbb{C}$. Moreover, $ap_j - \alpha_j = v_j + 2m_j\pi i$ for $j \in \{1, 2\}$ and some $m_1, m_2 \in \mathbb{Z}$. We must have $ap_j - v_j - 2m_j\pi i = \alpha_j \in \mathbb{R}$, $j \in \{1, 2\}$, that is,

$$\Im(ap_j) = \Im(v_j) + 2m_j\pi, \quad j \in \{1, 2\}.$$

□

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